

## COMMENT ON “ONE-WAY DEFICIT OF TWO QUBIT X STATES”

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ABSTRACT. We improve the recent method of Wang et. al to calculate exactly the one-way information deficit of any X-state. Analytical formulas of the one-way information deficit are given for several nontrivial regions of the parameters.

## 1. INTRODUCTION

Quantum discord is one of the quantum correlations that has been studied intensively similar to quantum entanglement in quantum information process. It measures the difference between the mutual information and maximal classical mutual information [1]. Besides the quantum discord, other quantum correlations have also been introduced in various applications [2, 3], for example, one-way information deficits. In [4, 5], Streltsov et al. defined the quantity by the relative entropy, which has shown its close relationship with quantum entanglement. The one-way information deficit by the Von Neumann measurement for a quantum density matrix  $\rho^{ab}$  with respect to the first component is given by [6]

$$(1.1) \quad \Delta^{\rightarrow}(\rho^{ab}) = \min_{\{P_i^a\}} S\left(\sum_i P_i^a \rho^{ab} P_i^a\right) - S(\rho^{ab}),$$

where the minimum is taken over Von Neumann measurements and  $S$  is the entropy. The Von Neumann measurements are parametrized by the set of all complete rank one orthogonal projectors  $P_i^a = |i^a\rangle\langle i^a|$  such that  $\sum_i P_i^a = I$ . In general it is a complex problem to compute the one-way information deficit exactly, similar to the situation of the quantum discord (cf. [7]).

Recently in [8], Wang et al. have proposed a method to evaluate the one-way information deficit for an  $X$  state, which generalizes the earlier work [9]. The idea of this new method is to reduce the calculation to an optimization question with fewer variables. However, we find that Ref. [8] provides the answer for degenerate cases and in general only gives an estimate. In this note, we generalize and improve their method and obtain an exact answer for the one-way information deficit of any  $X$ -state. Furthermore, several analytical formulas of the one-way information deficit are given for some nontrivial regions of the parameters, in particular, these include cases with nonzero  $z$ -components in the Bloch decomposition of the density matrix.

## 2. ONE-WAY INFORMATION DEFICIT FOR X STATES

First of all, one observes that the one-way information deficit is invariant under local unitary transformations. For a general two qubit  $X$ -state one considers the quantum state in the following form:

$$(2.1) \quad \rho^{ab} = \frac{1}{4}(I \otimes I + \sum_{i=1}^3 r_i \sigma_i \otimes I + I \otimes \sum_{i=1}^3 s_i \sigma_i + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i),$$

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where  $\mathbf{r} = \{r_1, r_2, r_3\}$ ,  $\mathbf{s} = \{s_1, s_2, s_3\}$  are the Bloch vectors,  $\sum_i r_i^2 \leq 1$ ,  $\sum_i s_i^2 \leq 1$ ,  $\sum_i c_i^2 \leq 1$  and  $\{\sigma_i\}_{i=1}^3$  are the standard Pauli spin matrices. Here the Bloch vectors are assumed in the  $z$ -direction, that is,  $\mathbf{r} = \{0, 0, r_3\}$ ,  $\mathbf{s} = \{0, 0, s_3\}$ . Then the state can be written as

$$(2.2) \quad \rho^{ab} = \frac{1}{4}(I \otimes I + r_3 \sigma_3 \otimes I + s_3 I \otimes \sigma_3 + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i),$$

where  $r_3, s_3 \in [-1, 1]$  and  $\sum_i c_i^2 \leq 1$ . In terms of the computational basis  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ , the matrix form of  $\rho^{ab}$  is

$$(2.3) \quad \rho = \frac{1}{4} \begin{pmatrix} 1 + r_3 + s_3 - c_3 & 0 & 0 & c_1 - c_2 \\ 0 & 1 + r_3 - s_3 - c_3 & c_1 + c_2 & 0 \\ 0 & c_1 + c_2 & 1 - r_3 + s_3 - c_3 & 0 \\ c_1 - c_2 & 0 & 0 & 1 - r_3 - s_3 + c_3 \end{pmatrix}.$$

The eigenvalues are:

$$\begin{aligned} \eta_{1,2} &= \frac{1}{4}(1 - c_3 \pm \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}), \\ \eta_{3,4} &= \frac{1}{4}(1 + c_3 \pm \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}). \end{aligned}$$

Then the entropy of  $\rho$  is given by

$$(2.4) \quad \begin{aligned} S(\rho) &= 2 - \frac{1}{4}(1 - c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \log_2(1 - c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \\ &\quad - \frac{1}{4}(1 - c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \log_2(1 - c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \\ &\quad - \frac{1}{4}(1 + c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \log_2(1 + c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \\ &\quad - \frac{1}{4}(1 + c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \log_2(1 + c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}). \end{aligned}$$

Note that any Von Neumann measurement can be realized by

$$(2.5) \quad \{B_k = V \Pi_k V^\dagger : k = 0, 1\}$$

where  $\Pi_k = |k\rangle\langle k| : k = 0, 1$  and  $V \in \text{SU}(2)$ . Then the minimum in Eq. (1.1) is taken over the group  $\text{SU}(2)$ . We will compute the one-way information deficit with respect to the second particle and the superscript of  $\Pi_k^b$  is removed for simplicity. Each unitary operator  $V \in \text{SU}(2)$  can be written as

$$V = tI + i \sum_{i=1}^3 y_i \sigma_i$$

where  $t, y_i \in \mathbb{R}$  and  $t^2 + \sum_{i=1}^3 y_i^2 = 1$ .

After the measurement  $\{B_k\}$ , the state  $\rho$  is changed to the ensemble  $\{\rho_k, p_k\}$ :

$$(2.6) \quad \rho_k = \frac{1}{p_k}(I \otimes B_k)\rho(I \otimes B_k),$$

where  $p_k = \text{tr}(I \otimes B_k)\rho(I \otimes B_k)$ . It follows from [8] that

$$\begin{aligned} p_0 \rho_0 &= \frac{1}{4}(I + s_3 z_3 I + r_3 \sigma_3 + \sum_{i=1}^3 (c_i z_i) \sigma_i) \otimes (V \Pi_0 V^\dagger), \\ p_1 \rho_1 &= \frac{1}{4}(I - s_3 z_3 I + r_3 \sigma_3 - \sum_{i=1}^3 (c_i z_i) \sigma_i) \otimes (V \Pi_1 V^\dagger), \end{aligned}$$

where  $z_1 = 2(y_1y_3 - ty_2)$ ,  $z_2 = 2(ty_1 + y_2y_3)$ ,  $z_3 = t^2 + y_3^2 - y_1^2 - y_2^2$ . Here it is easy to check directly that  $\sum_{i=1}^3 z_i^2 = 1$ . So the minimum in the one-way information deficit is taken over the unit sphere.

The eigenvalues of  $p_0\rho_0 + p_1\rho_1$  are

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{4} \left( 1 + s_3 z_3 \pm \sqrt{r_3^2 + 2r_3 z_3 + \sum_i (c_i z_i)^2} \right), \\ \lambda_{3,4} &= \frac{1}{4} \left( 1 - s_3 z_3 \pm \sqrt{r_3^2 - 2r_3 z_3 + \sum_i (c_i z_i)^2} \right).\end{aligned}$$

Let  $\phi = z_3$ ,  $\theta = c_1^2 z_1^2 + c_2^2 z_2^2 + c_3^2 z_3^2$ , then the entropy of  $\sum_k \Pi_k \rho \Pi_k$  is

$$\begin{aligned}(2.7) \quad S(\sum_k \Pi_k \rho \Pi_k) &= F(\theta, \phi) \\ &= 2 - \frac{1}{4} (1 + s_3 \phi + \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}) \log_2 (1 + s_3 \phi + \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}) \\ &\quad - \frac{1}{4} (1 + s_3 \phi - \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}) \log_2 (1 + s_3 \phi - \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}) \\ &\quad - \frac{1}{4} (1 - s_3 \phi + \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}) \log_2 (1 - s_3 \phi + \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}) \\ &\quad - \frac{1}{4} (1 - s_3 \phi - \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}) \log_2 (1 - s_3 \phi - \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}).\end{aligned}$$

Note that  $F(\theta, \phi)$  is an even function for the variable  $\phi$ , so we can focus on  $\phi \in [0, 1]$  instead of  $[-1, 1]$ .

As in [8], we compute that

$$\begin{aligned}(2.8) \quad \frac{\partial F}{\partial \theta} &= -\frac{1}{8} \frac{1}{\sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}} \log_2 \frac{1 + s_3 \phi + \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}}{1 + s_3 \phi - \sqrt{r_3^2 + 2r_3 c_3 \phi + \theta}} \\ &\quad - \frac{1}{8} \frac{1}{\sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}} \log_2 \frac{1 - s_3 \phi + \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}}{1 - s_3 \phi - \sqrt{r_3^2 - 2r_3 c_3 \phi + \theta}} < 0\end{aligned}$$

Thus  $S(\sum_k \Pi_k \rho \Pi_k) = F(\theta, \phi)$  is decreasing about  $\theta$ . For any fixed  $\phi = z_3 = \phi_0 \in [0, 1]$ , the minimum of  $F(\theta, \phi_0)$  should be achieved at the maximum allowable value of  $\theta$ . For  $\phi = \phi_0$ ,  $\theta = c_1^2 z_1^2 + c_2^2 z_2^2 + c_3^2 \phi_0^2$ , and since  $z_1^2 + z_2^2 + \phi_0^2 = 1$ , we get that

$$\begin{aligned}(2.9) \quad \theta &= c_1^2 z_1^2 + c_2^2 z_2^2 + c_3^2 \phi_0^2 \\ &\leq c^2 (z_1^2 + z_2^2) + c_3^2 \phi_0^2 \\ &= c^2 - c^2 \phi_0^2 + c_3^2 \phi_0^2 = c^2 + (c_3^2 - c^2) \phi_0^2,\end{aligned}$$

where  $c = \max\{|c_1|, |c_2|\}$ , and the equality can be achieved by appropriate  $t, y_i$  or  $z_1, z_2$ . In fact, if  $|c_1| \geq |c_2|$ , then  $c = |c_1|$ . Take  $z_2 = 0$ , then  $\theta = c^2 z_1^2 + c_3^2 \phi_0^2 = c^2 (1 - \phi_0^2) + c_3^2 \phi_0^2 = c^2 + (c_3^2 - c^2) \phi_0^2$ . So for each fixed  $\phi = \phi_0$  the maximum value of  $\theta$  is  $c^2 + (c_3^2 - c^2) \phi_0^2$ . Therefore for  $\phi \in [0, 1]$  the minimum of  $F(\theta, \phi)$ , or the maximum of  $2 - F(\theta, \phi)$ , is given by the maximum of the following

function

$$\begin{aligned}
 G(\phi) &= 2 - F(c^2 + (c_3^2 - c^2)\phi^2, \phi) \\
 (2.10) \quad &= \frac{1}{4}(1 + s_3\phi + R_+) \log_2(1 + s_3\phi + R_+) + \frac{1}{4}(1 + s_3\phi - R_+) \log_2(1 + s_3\phi - R_+) \\
 &\quad + \frac{1}{4}(1 - s_3\phi + R_-) \log_2(1 - s_3\phi + R_-) + \frac{1}{4}(1 - s_3\phi - R_-) \log_2(1 - s_3\phi - R_-),
 \end{aligned}$$

where  $R_{\pm} = \sqrt{r_3^2 \pm 2r_3c_3\phi + c^2 + (c_3^2 - c^2)\phi^2} = \sqrt{(r_3 \pm c_3\phi)^2 + c^2(1 - \phi^2)}$ . This transforms the question of  $\min S(\sum_k \Pi_k \rho \Pi_k)$  to the maximization of a one variable function.

By the definition of the one-way information deficit and Eq. (2.4) it follows that

$$\begin{aligned}
 \Delta^{\rightarrow}(\rho) &= \min_{\{B_k\}} S(\sum_k \Pi_k \rho \Pi_k) - S(\rho) \\
 (2.11) \quad &= \frac{1}{4}(1 - c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \log_2(1 - c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \\
 &\quad + \frac{1}{4}(1 - c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \log_2(1 - c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 + c_2)^2}) \\
 &\quad + \frac{1}{4}(1 + c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \log_2(1 + c_3 + \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \\
 &\quad + \frac{1}{4}(1 + c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \log_2(1 + c_3 - \sqrt{(r_3 - s_3)^2 + (c_1 - c_2)^2}) \\
 &\quad - \max_{\phi \in [0,1]} G(\phi),
 \end{aligned}$$

where  $c = \max\{|c_1|, |c_2|\}$  in the definition of  $G(\phi)$  (see Eq. (2.10)). Eq. (2.11) gives an exact answer of the one-way information deficit of an arbitrary X-state and replaces Eq. (15) of [9]. In particular, the formulas contain cases with nonzero components of the  $z$ -direction in the Bloch representation of the density matrix  $\rho$  (see Remark 2.3).

We now compute the maximum value of  $G(\phi)$  ( $0 \leq \phi \leq 1$ ) for some nontrivial cases. First we have

$$\begin{aligned}
 R'_{\pm}(\phi) &= R_{\pm}^{-1} \cdot (\pm r_3 c_3 + (c_3^2 - c^2)\phi), \\
 (2.12) \quad G'(\phi) &= \frac{R'_+}{4} \log_2 \frac{1 + s_3\phi + R_+}{1 + s_3\phi - R_+} + \frac{R'_-}{4} \log_2 \frac{1 - s_3\phi + R_-}{1 - s_3\phi - R_-} \\
 &\quad + \frac{s_3}{4} \log_2 \frac{(1 + s_3\phi + R_+)(1 + s_3\phi - R_+)}{(1 - s_3\phi + R_-)(1 - s_3\phi - R_-)}.
 \end{aligned}$$

Note that

$$(2.13) \quad R_+^2 - R_-^2 = 4r_3c_3\phi,$$

$$(2.14) \quad (1 + s_3\phi + R_+)(1 + s_3\phi - R_+) - (1 - s_3\phi + R_-)(1 - s_3\phi - R_-) = 4(s_3 - r_3c_3)\phi,$$

$$\begin{aligned}
 (2.15) \quad &(1 + s_3\phi - R_+)(1 - s_3\phi + R_-) - (1 + s_3\phi + R_+)(1 - s_3\phi - R_-) \\
 &= 2(R_- - R_+ + s_3\phi(R_+ + R_-)) = \frac{2\phi}{R_+ + R_-}(-2r_3c_3 + s_3(R_+ + R_-)^2)
 \end{aligned}$$

We then have the following results:

(i) Suppose  $c_3^2 - c^2 \geq r_3^2$ ,  $r_3c_3 \leq 0$ , and  $s_3 \geq 0$ . Then  $R'_- = R_-^{-1}(-r_3c_3 + (c_3^2 - c^2)\phi) \geq 0$  for  $\phi \in [0, 1]$ . By Eq. (2.14), the 3rd term in  $G'(\phi)$  in Eq. (2.12) is nonnegative. It follows from Eq. (2.15) that the first  $\log_2$ -expression of  $G'(\phi)$  is  $\leq$  the second one for  $\phi \in [0, 1]$ . Then we compute

that

$$\begin{aligned}
 G'(\phi) &\geq \frac{R'_+}{4} \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} + \frac{R'_-}{4} \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} \\
 (2.16) \quad &= \frac{1}{4} \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} \left( R'_+ + R'_- \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} \left( \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} \right)^{-1} \right) \\
 &\geq \frac{1}{4} \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} (R'_+ + R'_-).
 \end{aligned}$$

Since  $R'_+(0) + R'_-(0) = 0$  and

$$\begin{aligned}
 (2.17) \quad R''_+(\phi) + R''_-(\phi) &= \frac{d}{d\phi} \left( \frac{r_3 c_3 + (c_3^2 - c^2)\phi}{R_+} + \frac{-r_3 c_3 + (c_3^2 - c^2)\phi}{R_-} \right) \\
 &= c^2(c_3^2 - c^2 - r_3^2) \left( \frac{1}{R_+^3} + \frac{1}{R_-^3} \right),
 \end{aligned}$$

which is  $\geq 0$ , thus  $G'(\phi) \geq 0$ . Therefore the maximum of  $G(\phi)$  on  $[0, 1]$  is

$$\begin{aligned}
 (2.18) \quad G(1) &= \frac{1}{4} (1+s_3+|r_3+c_3|) \log_2 (1+s_3+|r_3+c_3|) \\
 &\quad + \frac{1}{4} (1+s_3-|r_3+c_3|) \log_2 (1+s_3-|r_3+c_3|) \\
 &\quad + \frac{1}{4} (1-s_3+|r_3-c_3|) \log_2 (1-s_3+|r_3-c_3|) \\
 &\quad + \frac{1}{4} (1-s_3-|r_3-c_3|) \log_2 (1-s_3-|r_3-c_3|).
 \end{aligned}$$

(ii) Suppose  $r_3 c_3 \geq 0$ ,  $c_3^2 - c^2 \geq r_3^2$ , and  $s_3 \leq 0$ . Now the 3rd term of  $G'(\phi)$  in Eq. (2.12) is still nonnegative, and  $R_+ \geq R_-$ . Note that  $R'_+ = R_+^{-1}(r_3 c_3 + (c_3^2 - c^2)\phi) \geq R_+^{-1} r_3 c_3 \geq 0$  for  $0 \leq \phi \leq 1$ , and the first log is  $\geq$  the second log in  $G'(\phi)$  by the first identity of Eq. (2.15). So

$$\begin{aligned}
 (2.19) \quad G'(\phi) &\geq \frac{R'_+}{4} \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} + \frac{R'_-}{4} \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} \\
 &= \frac{1}{4} \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} \left( R'_+ \log_2 \frac{1+s_3\phi+R_+}{1+s_3\phi-R_+} \left( \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} \right)^{-1} + R'_- \right) \\
 &\geq \frac{1}{4} \log_2 \frac{1-s_3\phi+R_-}{1-s_3\phi-R_-} (R'_+ + R'_-).
 \end{aligned}$$

Similarly by Eq. (2.17) we have  $R''_+(\phi) + R''_-(\phi) \geq 0$ , then  $G'(\phi) \geq 0$  for  $\phi \in [0, 1]$ . Therefore the maximum of  $G(\phi)$  on  $[0, 1]$  is again  $G(1)$  given in Eq. (2.18).

(iii) Suppose  $s_3 = r_3 c_3 \leq 0$ ,  $c^2 = c_3^2$ , and  $\max\{|c|, |r_3|\} \geq 1/\sqrt{2}$ . Now Eq. (2.14) implies that the third term of  $G'(\phi)$  is 0. Note that  $R'_+ = R_+^{-1}(r_3 c_3 + (c_3^2 - c^2)\phi) = R_+^{-1} r_3 c_3 \leq 0$ .

When  $c_3^2 = c^2$ , we have

$$\begin{aligned}
 (R_+ + R_-)^2 &= 2(r_3^2 + c^2 + (c_3^2 - c^2)\phi^2) + 2\sqrt{(r_3^2 + c^2 + (c_3^2 - c^2)\phi^2)^2 - 4r_3^2 c_3^2 \phi^2} \\
 &= 2(r_3^2 + c^2) + 2\sqrt{(r_3^2 + c^2)^2 - 4r_3^2 c_3^2 \phi^2} \\
 &\geq 2(r_3^2 + c^2 + |r_3^2 - c^2|) = 4 \max\{r_3^2, c^2\} \geq 2.
 \end{aligned}$$

So  $-2r_3c_3 + s_3(R_+ + R_-)^2 = s_3(-2 + (R_+ + R_-)^2) \leq 0$ , then it follows from Eq. (2.15) that the first-log expression  $\geq$  the second log-expression in  $G'(\phi)$ . Subsequently

$$\begin{aligned} G'(\phi) &= \frac{R'_+}{4} \log_2 \frac{1 + s_3\phi + R_+}{1 + s_3\phi - R_+} + \frac{R'_-}{4} \log_2 \frac{1 - s_3\phi + R_-}{1 - s_3\phi - R_-} \\ (2.20) \quad &\leq \frac{1}{4} \log_2 \frac{1 - s_3\phi + R_-}{1 - s_3\phi - R_-} (R'_+ + R'_-). \end{aligned}$$

Recall that  $R'_+(0) + R'_-(0) = 0$ , and Eq. (2.17) implies that  $R''_+(\phi) + R''_-(\phi) \leq 0$ , thus  $G'(\phi) \leq 0$  for  $\phi \in [0, 1]$ . Therefore the maximum of  $G(\phi)$  on  $[0, 1]$  is

$$(2.21) \quad G(0) = \frac{1}{2}(1 + \sqrt{r_3^2 + c^2}) \log_2(1 + \sqrt{r_3^2 + c^2}) + \frac{1}{2}(1 - \sqrt{r_3^2 + c^2}) \log_2(1 - \sqrt{r_3^2 + c^2}).$$

(iv) Suppose  $r_3 = 0$ . Then  $R_+ = R_-$ . It is easy to see that if  $s_3 \geq 0$  and  $c_3^2 \geq c^2$  (resp.  $s_3 \leq 0$  and  $c_3^2 \leq c^2$ ), then  $G(1)$  (resp.  $G(0)$ ) is the maximum:

$$\begin{aligned} G(1) &= \frac{1}{4}(1 + s_3 + |c_3|) \log_2(1 + s_3 + |c_3|) + \frac{1}{4}(1 + s_3 - |c_3|) \log_2(1 + s_3 - |c_3|) \\ &\quad + \frac{1}{4}(1 - s_3 + |c_3|) \log_2(1 - s_3 + |c_3|) + \frac{1}{4}(1 - s_3 - |c_3|) \log_2(1 - s_3 - |c_3|), \\ G(0) &= \frac{1}{2}(1 + |c|) \log_2(1 + |c|) + \frac{1}{2}(1 - |c|) \log_2(1 - |c|). \end{aligned}$$

We summarize the results in Table 1.

TABLE 1. Maximum value of  $G(\phi)$ .

Cases	Conditions				Maximum
(i)	$c_3^2 - c^2 \geq r_3^2$	$r_3c_3 \leq 0$	$s_3 \geq 0$		$G(1)$
(ii)	$c_3^2 - c^2 \geq r_3^2$	$r_3c_3 \geq 0$	$s_3 \leq 0$		$G(1)$
(iii)	$c_3^2 = c^2$		$s_3 = r_3c_3 \leq 0$	$\max\{ r_3 ,  c \} \geq \frac{\sqrt{2}}{2}$	$G(0)$
(iv)	$c_3^2 \geq c^2$	$r_3 = 0$	$s_3 \geq 0$		$G(1)$
	$c_3^2 \leq c^2$		$s_3 \leq 0$		$G(0)$

In the following we comment on our exact solutions.

*Remark 2.1.* In [8], it was argued that if one defines  $\phi = z_3$ ,  $\theta = c_1^2z_1^2 + c_2^2z_2^2 + c_3^2z_3^2$ , then  $\phi \in [-1, 1]$ ,  $\sqrt{\theta} \leq C = \max\{|c_1|, |c_2|, |c_3|\}$  and the equality can be achieved. We find that the maximum of  $\theta$  is only equal to  $C^2$  in degenerate cases. In fact, the variable  $\theta$  is not independent of  $\phi$ , and the maximum of  $\theta$  should be related with  $\phi$ . Actually following [8], since  $\theta = c_1^2z_1^2 + c_2^2z_2^2 + c_3^2z_3^2$ , to get the maximum value  $C^2$  means that  $(z_1, z_2, z_3)$  is fixed, then  $\phi = z_3$  is fixed and can not vary in the whole interval  $[-1, 1]$ .

For example, if  $|c_2| > |c_1|$ ,  $|c_2| > |c_3|$  and  $\theta = C^2$ , then we must have  $z_1 = z_3 = \phi = 0$  and  $z_2 = 1$ .

*Remark 2.2.* When  $r_3 = s_3 = 0$ , we have

$$\begin{aligned} G(\phi) &= \frac{1}{2}(1 + \sqrt{c^2 + (c_3^2 - c^2)\phi^2}) \log_2(1 + \sqrt{c^2 + (c_3^2 - c^2)\phi^2}) \\ &\quad + \frac{1}{2}(1 - \sqrt{c^2 + (c_3^2 - c^2)\phi^2}) \log_2(1 - \sqrt{c^2 + (c_3^2 - c^2)\phi^2}). \end{aligned}$$

Let  $C = \max\{|c_1|, |c_2|, |c_3|\}$ , it follows from the above table that

$$\max G(\phi) = \frac{1}{2}(1 + C) \log_2(1 + C) + \frac{1}{2}(1 - C) \log_2(1 - C).$$

Therefore, when  $r_3 = s_3 = 0$ ,

$$\begin{aligned} \Delta^\rightarrow(\rho) &= \min_{\{\Pi_k\}} S\left(\sum_k \Pi_k \rho \Pi_k\right) - S(\rho) \\ &= \frac{1}{4}(1 - c_1 + c_2 + c_3) \log_2(1 - c_1 + c_2 + c_3) + \frac{1}{4}(1 - c_1 - c_2 - c_3) \log_2(1 - c_1 - c_2 - c_3) \\ &\quad + \frac{1}{4}(1 + c_1 + c_2 - c_3) \log_2(1 + c_1 + c_2 - c_3) + \frac{1}{4}(1 + c_1 - c_2 + c_3) \log_2(1 + c_1 - c_2 + c_3) \\ &\quad - \frac{1}{2}(1 + C) \log_2(1 + C) - \frac{1}{2}(1 - C) \log_2(1 - C), \end{aligned}$$

which is the one-way information deficit of the Bell-diagonal state given in [9].

*Remark 2.3.* We have used  $C^{++}$  programs to check our results on randomly generated sets of parameters  $c_i, r_3, s_3$  such that  $\sum_i |c_i| < 1$  and  $|r_3| + |s_3| + |c_3| < 1$ . We have compared  $101 \times 101$  values of  $2 - F(\theta, \phi)$ , where  $\theta = c_1^2(1 - \phi^2) \cos^2 \alpha + c_2^2(1 - \phi^2) \sin^2 \alpha + c_3^2 \phi^2$  at the equal lattice points of  $(\alpha, \phi) \in [0, 2\pi] \times [0, 1]$  with 101 values of  $G(\phi)$  at the equal partition points of  $\phi$  on  $[0, 1]$  to find that their maximum values agree to our satisfaction.

We remark that with some simple choice of the 5 parameters such that  $r_3, s_3 \neq 0$ , Maple fails to compute the maximum values of  $G(\phi)$  and  $2 - F(\theta, \phi)$ .

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